Modeling LED street lighting

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LED luminaires may deliver precise illumination patterns to control light pollution, comfort, visibility, and light utilization efficiency. Here, we provide simple equations to determine how the light distributes in the streets. In particular, we model the illuminance spatial distribution as a function of Cartesian coordinates on a floor, road, or street. The equations show explicit dependence on the luminary position (pole height and arm length), luminary angle (fixture tilt), and the angular intensity profile (radiation pattern) of the LED luminary. To achieve this, we propose two mathematical representations to model the sophisticated intensity profiles of LED luminaries. Furthermore, we model the light utilization efficiency, illumination uniformity, and veiling luminance of glare due to one or several LED streetlamps. © 2014 Optical Society of America

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1. Introduction
The superior performance of light-emitting diode (LED) street illumination is changing the way we think about city lighting [1]. LED luminaires for road lighting have the potential to deliver precise beam patterns to minimize light pollution, increase comfort and visibility, and maximize efficiency by directing light to the appropriate area. To date, however, a photometric theory of light distribution in LED street lighting has not been reported (Fig. 1). In this paper, we provide a simplified theoretical platform for light management that provides guidance toward high-performance optical designs. We present a set of simple but accurate equations for photometric modeling.

To build the lighting model, we begin by analyzing two mathematical representations to model the sophisticated angular intensity patterns of LED luminaries. Then, we develop a simple equation of the illuminance spatial distribution as a function of the Cartesian coordinates on a road surface. It is an analytical model of the two-dimensional illuminance map on the street. The equation depends on key parameters, such as lamp height (pole), tilt (luminary angle), and the angular intensity profile of the luminary (radiation pattern). Finally, to show the potential of this theoretical framework, we model the illumination efficiency, illumination uniformity, and veiling luminance of glare due to one or several LED streetlamps.

2. Modeling the Radiation Pattern
The illumination properties of LED lamps are generally described by means of the radiation pattern, which is the angular intensity distribution or luminous flux in a given direction [2]. It is, perhaps, the most important characteristic of a lamp to be included in its light source model. In contrast to other
LED applications, LED street luminaries usually have sophisticated radiation patterns because they use optical components that greatly modify the distribution of the light emitted from each LED [3–7]. The resulting light distribution is difficult to model analytically because an LED luminary is an extended source. Fortunately, to model its radiation pattern, an LED luminary can be treated as a point source because it is located at a significant height from the ground [8,9].

In street lighting, goniometric diagrams are used to represent the radiation pattern, for a rapid interpretation of how light is distributed in all directions (Fig. 2). These diagrams represent a slice through the emission field and, thus, plot the intensity as a function of angular direction. The radiation pattern may be represented in polar [Fig. 2(a)] or Cartesian [Fig. 2(b)] diagrams. The polar curve shows how the light is distributed in all directions, and the Cartesian curve shows the precise shape of pattern. A simple way to guess how much the luminary light spreads in the street is through the viewing angle of luminous intensity (Fig. 2). We incorporate this parameter, the angle of peak intensity \( \theta_p \) (or total view angle \( 2\theta_p \)), in our modeling of the radiation pattern of LED luminaries.

In this section, we propose two mathematical representations to model the sophisticated intensity profiles of LED luminaries. In general, the radiation pattern of LED luminaries has a “whale tail” shape, which we classify as smooth and sharp intensity curves, which describes the intensity curve model.

A. Smooth Intensity Curves

Some LED luminaries emit light with a smooth angular variation, and then their radiation patterns may be mathematically represented by adding three Gaussian functions [2]. Therefore, the intensity \( I(\theta) \) [lumens/sr], defined as the luminous flux emitted per unit solid angle in a given direction \( \theta \), is

\[
I(\theta) = \sum_{i=1}^{3} g_{i1} \exp(-g_{i2}(|\theta| - g_{i3})^2),
\]

where \( \theta \) is the polar angle in a coordinate system centered in the LED lamp. Coefficients \( g_{i1} - g_{i3} \) depend on the specific shape of the radiation pattern. Equation (1) may be modeled in degrees or radians, depending on the \( g \) coefficients.

It is apparent from Eq. (1) that \( I(\theta) \) is a Gaussian-weighted sum of the luminary LED optics contributions over a set of emitting directions [2]. Equation (1) is widely applicable for many kinds of LED lamps without a high degree of edge steepness at the peak angle, which is related to \( g_{i3} \). Figure 3(a) shows the modeled radiation pattern of a major manufacturer. It is the intensity curve of an LED street luminary from CREE. The angular distribution reported by the manufacturer is also shown for comparison. Appendix A shows the coefficients of Eq. (1) used to plot Fig. 3(a).

B. Sharp Intensity Curves

High-performance LED luminaries emit light with a cut-off angular variation in order to put light only where it is necessary [3–7]. Then, such angular distributions show a discontinuity in the first derivative \( dI/d\theta \), at the peak angle \( \theta_p \) [10]. In this case, we mathematically represent the radiation distribution by adding two Gaussian functions and a step function, given by

\[
I(\theta) = UG(\theta) + (1 - U)G(\theta_p) \exp[-g_5(|\theta| - \theta_p)^2].
\]
where $\theta$ is the polar angle for a given direction and $\theta_p$ is the characteristic peak angle of sharp intensity curves. Function $G$ is a Gaussian, given by

$$G(\theta) = g_1 - g_2 \exp[-g_3(|\theta| - g_4)^2]. \quad (3)$$

and $U$ function is a simple step function, defined as

$$U = \begin{cases} 1 & \text{if } |\theta| < \theta_p \\ 0 & \text{if } |\theta| \geq \theta_p \end{cases}. \quad (4)$$

Coefficients $g_1$ to $g_6$ depend on the specific shape of the LED luminary emission.

We can interpret Eq. (2) as a consequence of the rapid falling of lighting at the boundaries of rectangular illumination distributions. As an ideal optical limit, a perfect rectangular illumination requires total steepness at the peak angle [10], but such intensity gradient is Gaussian-smoothed with real optics.

Now let us apply Eq. (2) for modeling LED luminaries of some major manufacturers. Figure 3(b) shows the modeled radiation pattern of the Golden Dragon with Argus lens lamp from OSRAM, where coefficients of Eq. (2) are $g_1 = 0.272$, $g_2 = -0.749$, $g_3 = 0.00107$, $g_4 = 75.19$, $g_5 = 0.017$, and $\theta_p = 70^\circ$. The plot also shows the curve reported in the manufacturer’s datasheet, for comparison. Figure 4 shows the modeled radiation pattern for a LD48 luminary of BBE LEDs. Figure 4(a) shows the intensity curve along the horizontal plane ($Y-Z$ plane), and Fig. 4(b) shows the modeled pattern in the vertical plane ($X-Z$ plane). To render this pattern, the coefficients of Eq. (2) are $g_1 = 1.515$, $g_2 = 1.14$, $g_3 = 0.0003$, $g_4 = 0$, and $g_5 = 0.015$, which are the same for both vertical and horizontal directions ($X-Z$ and $Y-Z$ planes). For $X-Z$ plane, $\theta_p = 23^\circ$ and for the $Y-Z$ plane $\theta_p = 51.5^\circ$. These coefficient values normalize the intensity and make Eq. (2) a function of angle $\theta$ in degrees.

C. Three-Dimensional Radiation Patterns

Although it is unusual that manufacturers report the 3D intensity curves, most of the LED luminaries are designed to have an asymmetric radiation pattern in two perpendicular azimuthal directions. The radiation pattern usually has a large peak angle along the roadway ($Y-Z$ plane), and a short peak angle through the cross section of the roadway ($X-Z$ plane). This is because light distributes better on the street area that is usually, but not always [7], rectangular-shaped. Therefore, to fully model the radiation pattern, an appropriate angular variation in three dimensions should be introduced.

I. Smooth 3D Radiation Patterns

In the case of smooth radiation patterns, a simple model is

$$I(\theta, \phi) = \sum_{i=1}^{3} g_{i1}(\phi) \exp[-g_{i2}(\phi)|\theta| - g_{i3}(\phi)^2]. \quad (5)$$

where $\theta$ is the polar angle and $\phi$ is the azimuthal angle in a spherical coordinate system centered in the luminary. Because Eq. (5) is intended for smooth
radiation patterns, dependence of functions $g_{i1}$–$g_{i3}$ on $\phi$ may be simply represented by a super-ellipse equation \cite{11}:

$$g_{ik}(\phi) = \frac{g_{x_{ik}}g_{y_{ik}}}{\sqrt{(g_{x_{ik}} \sin \phi)^{m_a} + (g_{y_{ik}} \cos \phi)^{m_a}}}.$$  \hspace{1cm} (6)

where $k = 1, 2, 3$. Coefficients $m_{ik}$, $g_{x_{ik}}$, and $g_{y_{ik}}$ depend on the specific shape of the radiation pattern. Here, $m_{ik}$ is an even integer, e.g., Eq. (6) becomes the ellipse equation in polar coordinates if $m_{ik} = 2$ (see Fig. 5). In most cases, only two or four of these azimuthal functions are needed to render the radiation pattern, and the remaining values of $g$ become constant, i.e., independent of $\phi$.

Equation (5) is very suitable for some LED luminaries because the Gaussian functions switch smoothly from vertical to horizontal azimuthal planes. The change between azimuthal profiles may be simplified because several of the equation parameters may be the same in the $X$–$Z$ and $Y$–$Z$ azimuthal planes.

2. Sharp 3D Radiation Patterns

High-performance LED luminaries emit light with a radiation pattern that illuminates a rectangular area to put light only where it is necessary \cite{3}. The radiation pattern is sharp; as that represented by Eq. (2), but the peak angle $\theta_p$ changes with the azimuthal direction. We therefore propose the following mathematical representation for the radiation pattern:

$$I(\theta, \phi) = UG(\theta) + (1 - U)G[\theta_p(\phi)]\exp\{-g_5(\phi)[|\theta| - \theta_p(\phi)]^2\},$$  \hspace{1cm} (7)

where $\theta$ and $\phi$ are the direction angles in spherical coordinates. Function $G(\theta)$ is given by Eq. (3), and function $g_5$ is

$$g_5(\phi) = \frac{g_{x_5}g_{y_5}}{\sqrt{(g_{x_5} \sin \phi)^2 + (g_{y_5} \cos \phi)^2}},$$  \hspace{1cm} (8)

where $g_{x_5}$ and $g_{y_5}$ are the parameters in the $X$–$Z$ and $Y$–$Z$ planes. The step function $U$ is

$$U = \begin{cases} 1 & \text{if } |\theta| < \theta_p(\phi) \\ 0 & \text{if } |\theta| \geq \theta_p(\phi) \end{cases}$$  \hspace{1cm} (9)

where the peak angle function $\theta_p(\phi)$ determines the rectangular shape of the illumination distribution. We derived a simple expression for $\theta_p(\phi)$ that assures the rectangular shape of the illuminance pattern, given by

$$\theta_p(\phi) = \arctan\left[\frac{\tan \theta_{px} \tan \theta_{py}}{\sqrt{(\tan \theta_{px} \sin \phi)^{m} + (\tan \theta_{py} \cos \phi)^{m}}}\right].$$  \hspace{1cm} (10)

where $\theta_{px}$ and $\theta_{py}$ are the peak angles in the $X$–$Z$ and $Y$–$Z$ planes, respectively, and $m$ is an even integer that gives the rectangular contour shape for $m \geq 4$.

Let us apply Eqs. (5) and (7) for modeling the luminaire datasheets of two major manufacturers. Figure 6 shows the modeled 3D intensity plot of both a BBE and an OSRAM LED streetlight source. In Fig. 6(a), the coefficients of Eq. (7) are $g_1 = 1.491$, $g_2 = 1.122$, $g_3 = 0.0003$, $g_4 = 0$, $g_{5x} = g_{5y} = 0.015$, which are the same for both the horizontal and vertical planes. The parameter of rectangular shape is $m = 6$. The parameters related to the planes $X$–$Z$ and $Y$–$Z$ are $\theta_{px} = 23^\circ$ and $\theta_{py} = 51.5^\circ$. These parameters make Eq. (7) a function of angle $\theta$ in degrees. Coefficients to model Fig. 6(b) by Eq. (5) are given in Appendix A.

3. Modeling the Illumination Distribution on the Street

The main photometric quantity for roadway lighting analysis and design calculations is the illuminance, which measures lumens per unit area (lm/m$^2$, footcandles, or lux) of light arriving at the road surface. The light distribution on the street may be modeled as the spatial variation $(x, y)$ of the illuminance, in which $x$ and $y$ are the spatial coordinates on the roadway [Fig. 7(a)].
For simplicity, let us consider that the LED luminary is, virtually always, so high that it may be considered a point source, and then the illuminance $E$ on the street is

$$E = \frac{I(\theta, \phi) \cos \theta_s}{r^2}. \quad (11)$$

where $I$ is the luminous intensity of the lamp, $r$ is the distance between the lamp and an illuminated point in $(\theta, \phi)$ direction, and $\theta_s$ is the angle subtended by the lamp and the normal of the floor at the illuminated point [see Fig. 7(a)].

### A. Two-Dimensional Illuminance Distribution

Let us begin with a two-dimensional case, a transverse cross section of the street in which the light rays all lie in the $X-Z$ plane. In the first case, we consider a simple street illumination setup, with one luminary, whose optical axis points into the floor without tilt, $\sigma = 0$ [see Fig. 7(b)]. In such a case, the illuminance as a function of the distance $x$ from the pole base, is

$$E(x) = \frac{I(\theta) \cos \theta_s}{h \left( x^2 + h^2 \right)^{3/2}}. \quad (12)$$

where $h$ is the mounting height. Function $I(\theta)$ is the angular intensity distribution of the luminary (lumens/sr), which may be modeled by Eqs. (1) or (2). Angle $\theta$ may be simply computed by $\theta = \arccos \left( \frac{h(x^2 + h^2)^{-0.5}}{h} \right)$.

If the luminary is attached to one arm to be displaced into the road by a distance $d$ [Fig. 7(b)], then the illuminance function may be calculated just by making $E(x - d)$ in Eq. (12).

In general, the 2D illuminance distribution, from a luminary inclined $\sigma$, and mounted in an arm with length $d$, is

$$E(x) = \frac{I(\theta - \sigma) \cos \theta_s}{\left( (x-d)^2 + h^2 \right)^{3/2}}. \quad (13)$$

where angle $\theta = \arccos \left( \frac{h(x-d)^2 + h^2} {h^2} \right)^{0.5}$. Equation (13) may be useful for rapid calculations across the street. Note that the origin of the coordinate system is located at the pole base, i.e., is at $x = 0$.

### B. Three-Dimensional Illuminance Distribution

Let us consider the three-dimensional case, in which the illuminance distribution of the entire street is obtained. In the case where the optical axis of lamp points into the floor ($\sigma = 0$), the illuminance in every point with coordinates $(x, y)$ is

$$E(x, y) = \frac{I(\theta, \phi) \cos \theta_s}{\left( x^2 + y^2 + h^2 \right)^{3/2}}. \quad (14)$$

where $I(\theta, \phi)$ is the 3D angular intensity distribution of the luminary (lumens/sr), which can be modeled by Eqs. (5) or (7). The radial angle $\theta$ may be simply computed by $\theta = \arccos \left( \frac{h(x^2 + y^2 + h^2)^{-0.5}}{h} \right)$.
computed by \( \theta = \arccos\left[\frac{h(x^2 + y^2 + h^2)^{0.5}}{h - d}\right] \) and the azimuthal angle by \( \phi = \arcsin\left[\frac{y(x^2 + y^2)^{0.5}}{h - d}\right] \).

If the luminary is attached to one arm or displaced \( d \) into the road and the luminary is tilted by \( \sigma \), then the illuminance function is

\[
E(x, y) = \frac{I(\theta, \phi)h}{[(x - d)^2 + y^2 + h^2]^{3/2}}, \tag{15}
\]

where \( I(\theta, \phi) \) is again the 3D angular intensity pattern given by Eqs. (5) or (7). But the dependence \((x, y)\) of direction angles changes with luminary inclination \( \sigma \); thus we derived an equation for \( \theta \) and \( \phi \).

The radial angle \( \theta \) is

\[
\theta = \arccos\left\{\frac{h \cos \sigma + (x - d) \sin \sigma}{\left[(x - d)^2 + y^2 + h^2\right]^{0.5}}\right\}, \tag{16}
\]

and the azimuthal angle \( \phi \) is

\[
\phi = \arcsin\left\{\frac{y}{\left[(x - d) \cos \sigma - h \sin \sigma]^2 + y^2\right]^{0.5}}\right\}. \tag{17}
\]

Note that the origin of the coordinate system is located at \((x = 0, y = 0)\).

Let us apply Eq. (15) for the luminaires with the radiation pattern shown in Fig. 6. Figures 8 and 9 show the modeled illuminance distribution. Figure 8 shows the illuminance plot through plane \( X-Y \) by Eq. (15) for the luminary emission of Fig. 6(a). The illumination pattern is shown for different lamp inclinations. The modeled illuminance at the roadway is illustrated in gray levels and in color graphs. Figure 9 shows the illuminance plot for the intensity pattern of Fig. 6(b). For visual display, each sub-figure shows a gray-level picture that is the gamma correction of Eq. (15), i.e., \( E(x, y)^{0.45} \). Each sub-figure is normalized to its maximum value of illuminance.

Finally, we compare our model with the most precise rival, the Monte Carlo ray tracing. Computations are performed for a novel luminaire, recently reported in [3]. Figure 10 shows the comparison, where (a) and (b) are the illuminance distributions on the street, produced by a luminary tilted by \( \sigma = 0^\circ \) and \( 15^\circ \), respectively. The realistic calculation is performed with ASAP software using Monte Carlo ray tracing to simulate the LED luminaire with all its optical components (see [3]). We assess the similarity between the ray-trace and the model calculation by using the normalized cross correlation (NCC) [12]. As the similarity increases, the NCC ranges from 0% to 100%, where 100% means that the illumination distributions are identical. Even though our model is a point source approximation [8, 9], the similarity is very high (see Fig. 10). Parameters for calculating the right-hand figures in (a) and (b) are: illumination area \( 14 \text{ m} \times 30 \text{ m} \), \( h = 10 \text{ m} \), \( d = 7 \text{ m} \) in (a), and \( d = 3.61 \text{ m} \) in (b). Figure 10 also shows,

![Fig. 8. Illuminance distribution on the street, modeled with Eq. (15) for different lamp tilts. Sub-figures (a)-(d) show light patterns for different luminary inclinations: \( \sigma = 0^\circ, 10^\circ, 15^\circ, \) and \( 25^\circ \), respectively. In each sub-figure, the left side shows a gray-level illuminance picture of area \( 10 \text{ m} \times 25 \text{ m} \), and a color plot is shown at the right, where the area enclosed by the red dotted rectangle is \( 10 \text{ m} \times 25 \text{ m} \). Parameters are: \( h = 6 \text{ m} \), \( d = 1.5 \text{ m} \), and intensity pattern of Fig. 6(a).](image-url)
in (c), the modeled radiation pattern using Eq. (7).
The coefficients required to model the radiation pattern with Eq. (7) are:
\( g_1 = 36.73 \), \( g_2 = 36.25 \),
\( g_3 = 7.237 \times 10^{-6} \), \( g_4 = 0 \), and \( g_5 = 0.027 \), which are azimuthally independent, and the only coefficients for the X–Z and Y–Z planes are:
\( \theta_{px} = 26^\circ \) and \( \theta_{py} = 44.5^\circ \). The parameter of the rectangular shape is \( m = 12 \).

Figures 8–10 illustrate the versatility of our light distribution model. In addition, the equations proposed may be suitable to introduce specifications and tolerances in \( d \), \( h \), \( \sigma \), and \( I(\theta, \phi) \). In addition, this model may help to select and set up LED luminaries in a street lighting system to meet specific performance criteria for efficiency and lighting quality [10].

4. Modeling Performance: Efficiency, Uniformity, and Glare
Street lighting performance is mainly assessed by both the quality and efficiency of light utilization [13–15]. It is then important to calculate the efficiency of light utilization, light pollution, the illumination uniformity, and the disability glare. Improving these lighting parameters ensures sustainable development, and improves the visual performance of drivers and pedestrians. The purpose of this section is to illustrate the potential of our model to assess LED street lighting performance.

A. Efficiency of Illumination and Light Pollution
Efficiency of illumination is the percentage of light emitted by the LED luminary that illuminates the
target area of the street. Since, typically, a considerable part of light falls outside of the roadway, the illumination efficiency becomes an important factor to evaluate the energy savings and lighting performance. Here, we consider the illumination efficiency as the ratio of the luminous flux falling on the target region of the street ($\Phi_{\text{street}}$) to the total luminous flux emitted by the luminary ($\Phi_{\text{luminary}}$):

$$\eta = \frac{\Phi_{\text{street}}}{\Phi_{\text{luminary}}} = \frac{\int E dA}{\int Id\Omega} = \frac{\int \int E(x,y) dxdy}{\int \int I(\theta, \phi) \sin \theta d\theta d\phi}. \quad (18)$$

where the relative illuminance $E$ [given by Eq. (14) or Eq. (15)] is integrated over the target area $A$. The area $A$ may be only the roadway, or slightly larger to include some light falling out of the roadway for pedestrian use. In general, however, the integrated area $A$ should be rectangular shaped. Due to the nature of LED sources, the intensity $I$ is integrated over the hemisphere $\Omega = \pi$ steradians ($0 \leq \theta \leq \pi/2$, $0 \leq \phi < 2\pi$). Note that, because Eq. (18) is a ratio, $E$ and $I$ may be given in absolute or relative units.

As much as illumination efficiency increases, light pollution decreases. Luminaries emitting much light in unwanted directions can obscure views of the stars, waste energy, and make it harder for drivers and pedestrian to see. There are three main consequences of light pollution [16]: sky glow, light trespass, and discomfort glare. Each of these effects is independent of the other, but the overall effect can be assessed by a light pollution ratio, which may be calculated by:

$$LP = \frac{\Phi_{\text{out-street}}}{\Phi_{\text{luminary}}} = \frac{\Phi_{\text{luminary}} - \Phi_{\text{street}}}{\Phi_{\text{luminary}}} = 1 - \eta. \quad (19)$$

Here, $\Phi_{\text{out-street}}$ is the luminous flux falling out of the target region of the street. Usually, for assessing light pollution, the target region $A$ is a rectangular region of finite size. Light pollution is that light falling in a virtual box enclosing the target region, consisting of a top–down plane and four vertical planes. The lumens falling on this virtual box are considered as pollutant lumens, contributing to sky glow because this light, either directly or through reflection, reaches the sky.

Let us apply Eq. (18) for the advanced LED luminaire modeled in Fig. 10. Figure 11(a) shows the illumination efficiency as a function of the inclination of the luminary, for three different luminary positions. The target area is of finite size $A = 14 \text{ m} \times 30 \text{ m}$, such that $1 - \eta$ may be regarded as the LP ratio.

B. Illumination Uniformity

Calculating how uniform the light distribution is throughout the street is important in lighting quality assessment. Perfect illumination uniformity means that the luminous flux is spatially invariant over the road or street. This is one of the prime lighting design considerations for a driver’s view while traveling along the road. The vision capabilities of drivers and walkers are increased under uniform illumination. While traversing their path of the road, drivers should be able to discern any people or animal crossing the road, as well as the traffic moving along the way. The driver’s vision mechanisms of object recognition are improved if the illumination distribution is homogenous along the way.

Uniformity metrics can retrieve illumination homogeneity information at different levels [17]. Most of the illumination uniformity metrics may be calculated using Eq. (14) or Eq. (15). However, there are some standards in street lighting for assessing how evenly light is distributed along the roadway, which are called uniformity ratios. Uniformity ratios measure proportions between the maximum, minimum, and average luminance or illuminance levels, on the roadway area or target region [13–15]. Overall, the uniformity ratio is given by [15]:

$$U_0 = \frac{E_{\text{min}}}{E_{\text{ave}}}. \quad (20)$$

where $E_{\text{min}}$ and $E_{\text{ave}}$ are the minimum and average illuminance on the target region, respectively. These values may be easily computed by $\min \{E(x,y)\}$ and average $\{E(x,y)\}$.

Figure 12 shows the calculation of $U_0$ on the street area illuminated by two adjacent luminaries. Figure 12(a) shows the evaluation of Eq. (20) for luminaries in a zig–zag arrangement. The analyzed LED luminary is of BBE LEDs, like that plotted in Fig. 8. The illumination uniformity is calculated as a function of the inclination of the luminary $\sigma$, and a function of distance between luminaries $s$. The area for calculating $U_0$ is the region between two adjacent luminaries, i.e., $10\text{ s}^2$. As an example, Fig. 12(b) shows the illuminance distribution for three pole separations. The illuminance distribution is modeled using Eq. (15), where the total illuminance is $E(x,y) + E(10 - x, y - s)$. 

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where $\theta_v$ is the angle (in degrees) between the luminaire and the line-of-sight of an observer [see Fig. 13(a)], which is usually regarded as parallel to the roadway or $y$ axis. $E_v$ is the veiling illuminance, which is the vertical illuminance in the eye of the observer. We may rewrite Eq. (21) as a function of the luminary intensity $I$:

$$L_v = 10 \frac{I \cos \theta_v}{r^2 (\theta_v^2 + 1.5 \theta_v^3)}.$$  

Then, the veiling luminance may be written as a function of Cartesian coordinates $(x, y)$ for a position on the street:

$$L_v(x, y) = \frac{10I(\theta, \phi) y (\theta_v^2 + 1.5 \theta_v)^{-1}}{[(x-d)^2 + y^2 + (h-h_v)^2]^{3/2}}.$$  

Here, $I(\theta, \phi)$ is the 3D angular intensity distribution of the luminary, which may be modeled by Eq. (5) or Eq. (7). The direction angles, $\theta$ and $\phi$, are those given by Eqs. (16) and (17). The veiling angle $\theta_v$ is

$$\theta_v = \arccos \left( \frac{|y|}{[(x-d)^2 + y^2 + (h-h_v)^2]^{1/2}} \right),$$

where $h_v$ is the eye height above the street, which is usually taken as 1.45 m [15].

Figure 13(b) shows the calculation of veiling luminance distribution at each observer position on the street. The observer stands at point $P(x, y)$ on the road, at the standard eye height above the road $h_v = 1.45$ m. Calculated light is due to three luminaries in the observer's field-of-view, i.e., $L_v(x, y) = L_v(x, y - 30) + L_v(x, y - 60)$, where pole separation is 30 m. The calculation is carried out for $h = 6$ m, $d = 1.5$ m, $\sigma = 15^\circ$, and the radiation pattern of Fig. 6(b). A spatial map of veiling luminance may help to design LED street lighting in such a way that

Fig. 12. Calculation of the illumination uniformity $U_v$ with luminaries in zig-zag configuration. (a) Illumination uniformity as a function of both luminary tilt $\sigma$ and pole distance $s$. (b) Illuminance distribution for the three points marked in (a). The red-dotted rectangle encloses the area for the $U_v$ calculation, as tens of m$^2$. The modeling is carried for $h = 6$ m, $d = 0$ m, and the intensity pattern of Fig. 6(a).

C. Disability Glare

The purpose of any luminary is to illuminate the street surface to make visible the environment. But the luminary light that directly hits the eye creates a glare, which is an unwanted effect in street lighting that may cause disability and discomfort to drivers and pedestrians. The loss of visual image contrast is a consequence of the veiling luminance which may cause disability and discomfort to drivers and pedestrians. The loss of visual image contrast is a consequence of the veiling luminance.

The luminary glare may cause a feeling of discomfort or disturbance in the visual field. The magnitude of glare may be estimated by calculating the equivalent veiling luminance. There are several metrics to assess the veiling luminance [13,15,18]. In general, this luminance is directly proportional to the luminaire lighting the human eye, and inversely proportional to the angular separation from the field-of-view. A commonly used expression is

$$L_v = 10 \frac{E_v}{\theta_v^2 + 1.5 \theta_v}.$$  

Fig. 13. Veiling luminance (glare) calculation. (a) Schematic of the geometry for calculating the veiling luminance at the observer position, standing on a point $P(x, y)$ on the road due to one luminary. (b) Veiling luminance, calculated by Eq. (23), at each observer's position $(x, y)$, due to three luminaries in the field-of-view.
drivers’ and pedestrians’ vision is not impaired by glare in their field-of-view [10].

This section illustrated the potential of our model to calculate the street lighting performance, which may be assessed by quality and efficiency parameters. Several other parameters are also important and their calculation may be approached by the mathematical model reported here. These parameters are well defined in standards that give the requirements for various classes of roads [13–15]. Important examples include: the surround ratio, which indicates how drivers see vehicles, pedestrians and animals on the side of the road for safety purposes; and the luminance, which determines how drivers and walkers see the light reflected by the surface of the street [19].

5. Summary
We have derived simple equations to determine how light is distributed in LED street lighting. First, we proposed two mathematical representations to model the sophisticated intensity profiles of LED luminaries: one equation for smooth intensity patterns, and one for intensity profiles with sharp peaks. Secondly, we modeled the illumination of the street, i.e., we determined equations to model the illumination distribution at any point on the floor, road, or street. Because the model is analytic, the equations show explicit dependence on key parameters like: the luminary height, arm length, luminary tilt, and the luminous intensity curve. Finally, we modeled the performance of LED luminaries. In particular, we calculated four important parameters of LED street lighting: light utilization efficiency, light pollution, illumination uniformity, and veiling luminance of glare. In general, the simplicity of the model makes it easy to understand, apply, and perhaps be enhanced. This model is the theoretical counterpart of Monte Carlo ray tracing and radiosity calculations [20], which may further improve the analysis and performance of LED street lighting systems. For example, this model may be useful to introduce random manufacturing and installation variations of LED luminaries to analyze and design street lighting systems. Future work may include deriving design conditions so that LED street lighting installations minimize light pollution, increase comfort and visibility, and maximize both illumination uniformity and light utilization efficiency.

Appendix A: Coefficients for Modeling the Radiation Pattern of LED Luminaries [Figs. 3(a) and 6(b)]

Coefficients of Eq. (1) for modeling Fig. 3(a) are:

\[ g_{x1} = 0.035, \quad g_{y1} = 0.4610, \quad g_{21} = 2.984, \]
\[ g_{x2} = 43, \quad g_{y2} = 61.81 \]
\[ g_{x3} = 0.035, \quad g_{y3} = 0.4738, \]
\[ g_{22} = 11.15, \quad g_{x32} = 43, \quad g_{y32} = 60.80 \]
\[ g_{13} = 0.1693, \quad g_{23} = 38.78, \quad g_{33} = 19.57 \]
\[ m_{11} = m_{12} = 8, \quad m_{31} = m_{32} = 2 \]

Note that the set of coefficients given normalizes the intensity equation to 1 at its maximum, and makes the equation a function of \( \theta \) in degrees. To compute luminous intensity in lumens/sr, just multiply \( I(\theta, \phi) \) by the maximum value of lm/sr or candelas.

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References

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20. For example, ASAP (http://www.breault.com/index.php), and DIALux (www.dial.de/DIAL/en/dialux.html) software.