Tercer Examen Parcial
Cálculo IV

Enero 2022

1. **Hallar el área del círculo D de radio R usando el teorema de Green.**

   Si C es una curva cerrada simple que acota la región D:

$$\int_C xdy = \int_D \frac{\partial}{\partial x}(x)dA = \int_D dA = A$$

$$\int_C (-y)dy = \int_D -\frac{\partial}{\partial y}(-y)dA = \int_D dA = A$$

Entonces:

$$\frac{1}{2}\int_C xdy + (-y)dx = \frac{1}{2}(A + A) = \frac{1}{2}(2A) = A$$

$$A = \frac{1}{2}\int_C xdy - ydx$$

Como D es un círculo de radio R:

$$x = R\cos\theta$$

$$y = R\sin\theta$$

Sustituyendo:

$$A = \frac{1}{2}\int_0^{2\pi} (R\cos\theta)(R\cos\theta d\theta) - (R\sin\theta)(-R\sin\theta d\theta)$$

$$A = \frac{1}{2}\int_0^{2\pi} R^2\cos^2\theta d\theta + R^2\sin^2\theta d\theta = \frac{1}{2}\int_0^{2\pi} R^2(\cos^2\theta + \sin^2\theta)d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} R^2 d\theta = \frac{1}{2}R^2[\theta]_0^{2\pi} = \frac{1}{2}R^2(2\pi - 0) = \pi R^2$$

2. **Use el teorema de Stokes con $\vec{F} = (x^2+y-4)\hat{\imath}+3xy\hat{\jmath}+(2xz+z^2)$ y S es $x^2+y^2+z^2 = 16$ con $z \geq 0$.**

   Primero, encontremos el rotacional de $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$\left(\nabla \times \vec{F}\right)_x = \frac{\partial}{\partial y}(2xz+z^2) - \frac{\partial}{\partial z}(3xy) = 0$$

$$\left(\nabla \times \vec{F}\right)_y = -\left[\frac{\partial}{\partial x}(2xz+z^2) - \frac{\partial}{\partial z}(x^2+y-4)\right] = -(2z) = -2z$$

$$\left(\nabla \times \vec{F}\right)_z = \frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(x^2+y-4) = 3y-1$$

$$\nabla \times \vec{F} = -2z\hat{\jmath} + (3y-1)\hat{k}$$

   Ahora, considerando que:

$$x = 4\cos\theta\sin\phi$$

$$y = 4\sin\theta\sin\phi$$

$$z = 4\cos\phi$$

   Definamos el siguiente vector:

$$\vec{r} = (x, y, z) = (4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi)$$

   Ahora determinemos los siguientes vectores:

$$\vec{r}_\theta = \frac{\partial\vec{r}}{\partial\theta} = \left[\frac{\partial}{\partial\theta}(4\sin\phi\cos\theta), \frac{\partial}{\partial\theta}(4\sin\phi\sin\theta), \frac{\partial}{\partial\theta}(4\cos\phi)\right]$$

$$\vec{r}_\phi = \frac{\partial\vec{r}}{\partial\phi} = \left[\frac{\partial}{\partial\phi}(4\sin\phi\cos\theta), \frac{\partial}{\partial\phi}(4\sin\phi\sin\theta), \frac{\partial}{\partial\phi}(4\cos\phi)\right]$$

Al realizar las debidas derivadas parciales:

$$\vec{r}_\theta = (-4\sin\phi\sin\theta, 4\sin\phi\cos\theta, 0)$$

$$\vec{r}_\phi = (4\cos\phi\cos\theta, 4\cos\phi\sin\theta, -4\sin\phi)$$

Ahora, realicemos el siguiente producto cruz:

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \end{vmatrix}$$

$$\vec{r}_\theta \times \vec{r}_\phi = (-16\sin^2\phi\cos\theta)\hat{i} - (16\sin^2\phi\sin\theta)\hat{j} + (-16\sin\phi\cos\phi\sin^2\theta - 16\sin\phi\cos\phi\cos^2\theta)\hat{k}$$

$$\vec{r}_\theta \times \vec{r}_\phi = -(16\sin^2\phi\cos\theta)\hat{i} - (16\sin^2\phi\sin\theta)\hat{j} - \left[16\sin\phi\cos\phi(\sin^2\theta + \cos^2\theta)\right]\hat{k}$$

$$\vec{r}_\theta \times \vec{r}_\phi = -(16\sin^2\phi\cos\theta)\hat{i} - (16\sin^2\phi\sin\theta)\hat{j} - (16\sin\phi\cos\phi)\hat{k}$$

$$\vec{r}_\theta \times \vec{r}_\phi = -4\sin\phi\left[(4\sin\phi\cos\theta)\hat{i} + (4\sin\phi\sin\theta)\hat{j} + (4\cos\phi)\hat{k}\right]$$

Pero si observamos con detenimiento, podemos hacer la siguiente sustitución:

$$\vec{r}_\theta \times \vec{r}_\phi = -(4\sin\phi)\vec{r}$$

Aunque, para este problema en particular ocupamos que el vector radial siempre apunte hacia afuera, por lo que:

$$\vec{r}_\phi \times \vec{r}_\theta = (4\sin\phi)\vec{r}$$

Con lo obtenido anteriormente, podemos realizar la siguiente integral:

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} \left(\nabla \times \vec{F}\right) \cdot (4\sin\phi)\vec{r}\,d\phi d\theta$$

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 4\sin\phi\,(0, -2z, 3y-1) \cdot (x, y, z)\,d\phi d\theta$$

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 4\sin\phi\,(-2zy + 3yz - z)\,d\phi d\theta$$

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 4\sin\phi\,(yz - z)\,d\phi d\theta$$

Ajustando el término $yz - z$ a términos de ϕ y θ:

$$yz - z = (4\sin\theta\sin\phi)(4\cos\phi) - (4\cos\phi) = 16\sin\theta\sin\phi\cos\phi - 4\cos\phi$$

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 4\sin\phi\,(16\sin\theta\sin\phi\cos\phi - 4\cos\phi)\,d\phi d\theta$$

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} \left(62\sin\theta\sin^2\phi\cos\phi - 16\sin\phi\cos\phi\right) d\phi d\theta$$

Separando en dos integrales:

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} \left(62\sin\theta\sin^2\phi\cos\phi\right) d\phi d\theta - \int_0^{2\pi}\int_0^{\pi/2} (16\sin\phi\cos\phi)\,d\phi d\theta$$

Resolviendo la integral izquierda:

$$\int_0^{2\pi}\int_0^{\pi/2} \left(62\sin\theta\sin^2\phi\cos\phi\right) d\phi d\theta = 62\int_0^{2\pi} (\sin\theta)\,d\theta \int_0^{\pi/2} \left(\sin^2\phi\cos\phi\right) d\phi$$

$$\int_0^{2\pi}\int_0^{\pi/2} \left(62\sin\theta\sin^2\phi\cos\phi\right) d\phi d\theta = 62\,[\cos\theta]_{2\pi}^{0} \int_0^{\pi/2} \left(\sin^2\phi\cos\phi\right) d\phi$$

$$\int_0^{2\pi}\int_0^{\pi/2} \left(62\sin\theta\sin^2\phi\cos\phi\right) d\phi d\theta = 62\,[\cos(0) - \cos(2\pi)] \int_0^{\pi/2} \left(\sin^2\phi\cos\phi\right) d\phi$$

$$\int_0^{2\pi}\int_0^{\pi/2} \left(62\sin\theta\sin^2\phi\cos\phi\right) d\phi d\theta = 62\,(1-1) \int_0^{\pi/2} \left(\sin^2\phi\cos\phi\right) d\phi = 0$$

Resolviendo la integral derecha:

$$\int_0^{2\pi}\int_0^{\pi/2} (16\sin\phi\cos\phi)\,d\phi d\theta = 16\int_0^{2\pi} d\theta \int_0^{\pi/2} (\sin\phi\cos\phi)\,d\phi = 16\,[\theta]_0^{2\pi} \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2}$$

$$\int_0^{2\pi}\int_0^{\pi/2} (16\sin\phi\cos\phi)\,d\phi d\theta = 16\,(2\pi - 0)\left[\frac{1}{2}\sin^2\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin^2(0)\right] = 16(2\pi)\left(\frac{1}{2}\right)$$

$$\int_0^{2\pi}\int_0^{\pi/2} (16\sin\phi\cos\phi)\,d\phi d\theta = 16\pi$$

Reemplazando en la integral original:

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = 0 - 16\pi = -16\pi$$

Usando el teorema de Stokes:

$$\oint_C \vec{F} \cdot d\vec{s} = \int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = -16\pi$$

3. **Sea** $\vec{F} = (ye^z)\hat{i} + (xe^z)\hat{j} + (xye^z)\hat{k}$**, muestre que**

$$\int_C \vec{F} \cdot d\vec{s} = 0$$

Primero debemos determinar $\nabla \times \vec{F}$*:*

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix}$$

$$\left(\nabla \times \vec{F}\right)_x = \frac{\partial}{\partial y}(xye^z) - \frac{\partial}{\partial z}(xe^z) = xe^z - xe^z = 0$$

$$\left(\nabla \times \vec{F}\right)_y = -\left[\frac{\partial}{\partial x}(xye^z) - \frac{\partial}{\partial z}(ye^z)\right] = -\left(ye^z - ye^z\right) = 0$$

$$\left(\nabla \times \vec{F}\right)_z = \frac{\partial}{\partial x}(xe^z) - \frac{\partial}{\partial y}(ye^z) = e^z - e^z = 0$$

Por ende concluimos que $\nabla \times \vec{F} = \vec{0}$*. Usando el teorema de Stokes:*

$$\int_C \vec{F} \cdot d\vec{s} = \int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \int_S \vec{0} \cdot d\vec{S} = 0$$

4. **Sea** $\vec{F} = y\hat{i} + [z\cos(yz) + x]\hat{j} + y\cos(yz)\hat{k}$**, determine** f **tal que** $\vec{F} = \nabla f$**.**

Primero debemos determinar si $\vec{F}$ *es irrotacional. Para ello:*

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z\cos(yz) + x & y\cos(yz) \end{vmatrix}$$

$$\left(\nabla \times \vec{F}\right)_x = \frac{\partial}{\partial y}[y\cos(yz)] - \frac{\partial}{\partial z}[z\cos(yz) + x]$$

$$\left(\nabla \times \vec{F}\right)_y = -\left\{\frac{\partial}{\partial x}[y\cos(yz)] - \frac{\partial}{\partial z}(y)\right\}$$

$$\left(\nabla \times \vec{F}\right)_z = \frac{\partial}{\partial x}[z\cos(yz) + x] - \frac{\partial}{\partial y}(y)$$

Realizando las respectivas derivadas parciales:

$$\left(\nabla \times \vec{F}\right)_x = \cos(yz) - yz\sin(yz) - \cos(yz) + zy\sin(yz) = 0$$

$$\left(\nabla \times \vec{F}\right)_y = 0$$

$$\left(\nabla \times \vec{F}\right)_z = 1 - 1 = 0$$

Concluimos que $\nabla \times \vec{F} = \vec{0}$*, lo cual indica que* $\vec{F}$ *es irrotacional. Ahora, determinaremos* f *con la siguiente expresión:*

$$f = \int_0^x F_1(t,0,0)dt + \int_0^y F_2(x,t,0)dt + \int_0^z F_3(x,y,t)dt$$

Donde $F_1 = y$*,* $F_2 = z\cos(yz) + x$ *y* $F_3 = y\cos(yz)$*. Sustituyendo:*

$$f = \int_0^x (0)dt + \int_0^y [(0)\cos(0) + x]dt + \int_0^z [y\cos(yt)]dt = \int_0^y xdt + \int_0^z [y\cos(yt)]dt$$

Resolviendo cada integral:

$$f = x[t]_0^y + [\sin(yt)]_0^z = x(y-0) + [\sin(yz) - \sin(0)] = xy + \sin(yz)$$

5. **Sea** $\vec{F} = (x^3)\hat{\imath} + (y^3)\hat{\jmath} + (z^3)\hat{k}$**. Evaluar la integral de superficie sobre la esfera unidad.**

   Primero debemos determinar $\nabla \cdot \vec{F}$*:*

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2) = 1$$

Dado que la esfera unidad es $x^2 + y^2 + z^2 = 1$*. Ahora, usando la ley de Gauss:*

$$\int_S \vec{F}\cdot d\vec{S} = \int_V \left(\nabla \cdot \vec{F}\right) dV = \int_V (3)\, dV = 3\int_V dV$$

Como V *representa el volumen de una esfera de radio* $R = 1$*:*

$$\int_S \vec{F} \cdot d\vec{S} = 3\left(\frac{4}{3}\pi\right) = 4\pi$$